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Automorphisms of Subgraphs Obtained by Deleting a Pendant Vertex

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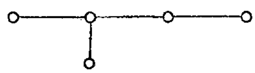
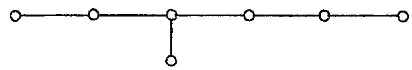
A pendant vertex, x, of a finite graph, G, is $*$-fixed in case every automorphism of $G - x$ fixes the unique vertex of G adjacent to x. It is proved that "almost all" finite graphs which have pendant vertices have $*$-fixed pendant vertices. In addition finite trees which have $*$-fixed pendant vertices are characterized.

Let G denote a finite graph without loops or multiple edges and let $V(G)$, $E(G)$, and $A(G)$ denote, respectively, the vertex set, edge set, and automorphism group of G. Let $x \in V(G)$. The valency of x in G is denoted with $\text{val}(G, x)$ and x is defined to be *pendant* in G in case $\text{val}(G, x) = 1$. The subgraph of G obtained by deleting x and all edges incident to x is denoted with $G - x$. If x is pendant in G then x^* denotes the unique vertex of G adjacent to x and x is said to be $*$-*fixed* whenever x^* is fixed by every automorphism of $G - x$. Intuitively one may think of a pendant vertex as being $*$-fixed whenever its removal does not add any symmetry to G.

A finite tree is *non-trivial* whenever it has at least two vertices. Henceforth let U_1 and U_2 denote the trees represented in figures A and B, respectively. A finite tree is *unusual* in case it is either an arc of length at least 2 ("arc" is as in [3]) or is isomorphic with U_1 or U_2.



A



B

THEOREM 1. *A non-trivial finite tree has a $*$-fixed pendant vertex if and only if it is not unusual.*

COROLLARY 1. *If T is a non-trivial finite asymmetric tree, then there exist asymmetric subtrees $\{T_i : 1 \leqslant i \leqslant k\}$ such that $T_1 = T$, T_k is isomorphic with U_2 and there exists, for every $i = 1, 2,\ldots, k$, a pendant vertex y_i of T_i satisfying $T_{i+1} = T_i - y_i$.*

Corollary 1 follows immediately from Theorem 1 whose proof appears below. Corollary 1 has been independently discovered and utilized by at least two other authors (cf. [1] and [2]). Theorems 1 and 2 have appeared in [4], where the proofs are much more complicated than those given here.

A *pruning* of a finite graph G is a decomposition:

$$G = G^{\natural} \cup T_1 \cup T_2 \cup \cdots \cup T_k, \tag{1}$$

where $G^{\natural}$ is the maximal subgraph of G each vertex of which has valency at least 2 and where $\{T_i : 1 \leqslant i \leqslant k\}$ is a non-empty set of disjoint non-trivial trees each having exactly one vertex in common with $G^{\natural}$. It is easily verified that G has a pruning if and only if G has a pendant vertex and no component of G is a tree. G is *domesticated* in case G has a pruning (1) such that $|V(T_i)| \geqslant 3$ $(1 \leqslant i \leqslant k)$. Theorem 2, which appears immediately below, when taken together with Theorem 1, indicates that "almost all" finite graphs which have a pendant vertex have a $*$-fixed pendant vertex. The proof of Theorem 2 appears at the end of this paper.

THEOREM 2. *A domesticated finite graph has a $*$-fixed pendant vertex.*

These results depend heavily on simple properties of trees. To improve them by considering all finite graphs G which have a pruning (1) necessitates arguments involving the automorphism group of $G^{\natural}$ which, as is well known, can be isomorphic with an arbitrary finite group. Nonetheless the authors wish to suggest the following conjecture as a feasible research problem:

CONJECTURE. *If G is a finite graph having a pruning* (1) *such that*

$$\max\{|E(T_i)| : 1 \leqslant i \leqslant k\} > \max\{\mathrm{val}(G^{\natural}, x) : x \in V(G^{\natural})\},$$

then G has a $*$-fixed pendant vertex.

Henceforth let T denote a non-trivial finite tree. The *distance* between two vertices, x and y, of T is the number of edges in the unique arc of T having x and y as end vertices. Let T'' denote the intersection of all longest arcs in T. Let T' denote the *center* of T, i.e., the largest subgraph whose vertex set consists of those vertices, z, such that the maximum distance between z and a pendant vertex is smallest possible. For every $x \in V(T)$ let T_x denote the largest connected subgraph of T which contains x and exactly one vertex of T'. If $|V(T')| = 1$ then $T_x = T$ and if $|V(T')| = 2$ then there exist vertices, u and v, such that $T = T_u \cup T' \cup T_v$. The following two lemmas are well known and easily verified:

LEMMA 1. *If $g \in A(T)$, then $gT' = T'$ and $gT'' = T''$.*

LEMMA 2. *If t is a pendant vertex of T, then $t \in V(T'')$ if and only if $T' \neq (T - t)'$. If, also, T has at least 3 vertices and $t \in V(T'')$, then either T' consists of two vertices, x which is the unique vertex of $(T - t)'$ and y which is closer to t than is x, or $(T - t)'$ consists of two vertices, x which is the unique vertex of T' and y which is further from t than is x.*

Henceforth let

$$P = (a_0, A_1, a_1, \ldots, A_n, a_n)$$

denote a simple (i.e., "non-self-intersecting") path in T. For every edge E of T let $C_P(E)$ denote that component of the graph obtained by removing E from T which does not contain a_0. P is said to be *a_0-peripheral* in case a_n is pendant in T and, for every $i = 1, 2, \ldots, n$, $C_P(A_i)$ has fewest vertices possible. P is said to be *$*$-radial* in case $a_0 \in V(T')$, a_n is pendant in T, and for every $i = 1, 2, \ldots, n$, either $A_i \in E(T) \setminus E(T'')$ and $C_P(A_i)$ has fewest vertices possible or, if no such A_i exists, $A_i \in E(T'') \setminus E(T')$ and $C_P(A_i)$ has fewest vertices possible.

LEMMA 3. *If P is $*$-radial or a_0-peripheral and there exists $g \in A(T - a_n)$ with $ga_0 = a_0$, then $ga_{n-1} = a_{n-1}$.*

Proof. Assume $ga_{n-1} \neq a_{n-1}$ and let j be the least positive integer such that $ga_j \neq a_j$. Then $n \geqslant 2$, A_j and gA_j are distinct edges of $T - a_n$

incident with a_{j-1}, and $a_0, a_1, \ldots, a_{j-1}$ are fixed by g. Hence $ga_j \notin V(P)$ and

$$g(C_P(A_j) - a_n) = C_P(gA_j). \tag{2}$$

This means that P cannot be a_0-peripheral, since this would require that $|V(C_P(A_j))| \leqslant |V(C_P(gA_j))|$. Therefore, P is $*$-radial, $a_0 \in V(T')$, and $a_1 \notin V(T')$. If, also, $ga_j \in V(T')$, then T' consists of two vertices, a_0 and ga_j, joined by an edge. Hence, $j = 1$ and, by Lemma 2, $ga_1 \in V((T - a_n)')$. Thus, by Lemma 1, $a_1 \in V((T - a_n)')$, contradicting $(T - a_n)' \subseteq T'$ so that $ga_j \notin V(T')$. But now, since P is $*$-radial, it follows from (2) that $gA_j \in E(T'')$ and $A_j \notin E(T'')$. Thus $a_n \notin V(T'')$, and so $T'' = (T - a_n)''$. But, by Lemma 1, it is not possible for both $gA_j \in E((T - a_n)'')$ and $A_j \notin E((T - a_n)'')$ to hold, a contradiction proving the lemma. ∎

Proof of Theorem 1. Unusual trees clearly have no $*$-fixed pendant vertex. To prove the converse assume that T is a non-trivial finite tree with no $*$-fixed pendant vertex. Then T has at least three vertices, and so if T is an arc then T is unusual. Suppose that T is not an arc. Then T has a $*$-radial path, P, with $a_n \notin V(T'')$. By Lemma 2, $(T - a_n)' = T'$. Since a_n is not $*$-fixed it follows from Lemma 3 that there exists $g \in A(T - a_n)$ such that $ga_0 \neq a_0$. Therefore,

$$V(T') = V((T - a_n)') = \{a, b\},$$

where $a = a_0$ and $b = ga_0$. Further, $T = T_a \cup T' \cup T_b$, $T_b = g(T_a - a_n)$, and T_b has at least two vertices. Let Q be a $*$-radial path with ends b and p where p is a pendant vertex of T contained in $V(T_b)$. Again p is not $*$-fixed so that there exists $h \in A(T - p)$ such that $hb \neq b$. If $p \notin V(T'')$, then $a = hb$ and $T_a = h(T_b - p)$, which is absurd because $T_b = g(T_a - a_n)$. Therefore, $p \in V(T'')$, and so by definition of a $*$-radial path $T' \cup T_b$ is an arc. Consequently,

$$T - a_n = (T_a - a_n) \cup T' \cup T_b$$

is an arc in T. Since $a_n \notin V(T'')$, $T - a_n$ is a maximum length arc in T, and so $a_n{}^*$ is the only vertex of T having valency 3. Because no pendant vertex of T is $*$-fixed it follows that T is isomorphic with U_1 or U_2. ∎

Proof of Theorem 2. Let (1) be a pruning of G, T be a T_i with fewest vertices, a_0 be the unique vertex both in T and $G^{\natural}$, and P be an a_0-peripheral path in T. Since G is domesticated, $|V(T - a_n)| \geqslant 2$, and so $ga_0 = a_0$ for every $g \in A(G - a_n)$. Therefore, by Lemma 3, a_n is $*$-fixed. ∎

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